



Integral Representation of Continuous Operators with Respect to Strict Topologies

Marian Nowak

Abstract. Let X be a completely regular Hausdorff space and $\mathcal{B}o$ be the σ -algebra of Borel sets in X . Let $C_b(X)$ (resp. $B(\mathcal{B}o)$) be the space of all bounded continuous (resp. bounded $\mathcal{B}o$ -measurable) scalar functions on X , equipped with the natural strict topology β . We develop a general integral representation theory of (β, ξ) -continuous operators from $C_b(X)$ to a lcHs (E, ξ) with respect to the representing Borel measure taking values in the bidual E''_ξ of (E, ξ) . It is shown that every (β, ξ) -continuous operator $T : C_b(X) \rightarrow E$ possesses a (β, ξ_ε) -continuous extension $\hat{T} : B(\mathcal{B}o) \rightarrow E''_{\xi'}$, where ξ_ε stands for the natural topology on E''_ξ . If, in particular, X is a k -space and (E, ξ) is quasicomplete, we present equivalent conditions for a (β, ξ) -continuous operator $T : C_b(X) \rightarrow E$ to be weakly compact. As an application, we have shown that if X is a k -space and a quasicomplete lcHs (E, ξ) contains no isomorphic copy of c_0 , then every (β, ξ) -continuous operator $T : C_b(X) \rightarrow E$ is weakly compact.

Mathematics Subject Classification. 46G10, 28A32, 28A25, 46A70.

Keywords. Spaces of bounded continuous functions, radon measures, vector measures, strict topologies, weakly compact operators, integration operators.

1. Introduction and Preliminaries

We assume that (E, ξ) is a locally convex Hausdorff space (briefly, lcHs) over either the complex field, \mathbb{C} , or the real field, \mathbb{R} . By $(E, \xi)'$ or E'_ξ we denote the topological dual of (E, ξ) . By $\sigma(L, K)$, $\beta(L, K)$ and $\tau(L, K)$ we denote weak topology, the strong topology and the Mackey topology on L with respect to a dual pair $\langle L, K \rangle$, respectively.

Throughout the paper we assume that (X, \mathcal{T}) is a completely regular Hausdorff space. By \mathcal{K} we will denote the family of all compact sets in X . Let

$\mathcal{B}o$ stand for the σ -algebra of Borel sets in X . Let $C_b(X)$ (resp. $B(\mathcal{B}o)$) be the Banach space of all bounded continuous (resp. bounded $\mathcal{B}o$ -measurable) scalar functions on X , equipped with the topology τ_u of the uniform norm $\|\cdot\|_\infty$. Let $C_b(X)'$ stand for the Banach dual of $C_b(X)$, equipped with the conjugate norm $\|\cdot\|'$. By $\mathcal{S}(\mathcal{B}o)$ we denote the space of all $\mathcal{B}o$ -simple functions on X .

Recall that a linear operator T from the Banach space $C_b(X)$ (resp. $B(\mathcal{B}o)$) to a lchS (E, ξ) is said to be *weakly compact* if T maps τ_u -bounded sets in $C_b(X)$ (resp., $B(\mathcal{B}o)$) onto relatively $\sigma(E, E'_\xi)$ -compact sets in E .

Following [14] the *strict topology* β on $B(\mathcal{B}o)$ is defined by the family of seminorms

$$p_w(v) := \sup_{t \in X} w(t)|v(t)| \quad \text{for } v \in B(\mathcal{B}o),$$

where w runs over the family $B_o(X)^+$ of all bounded functions $w : X \rightarrow [0, \infty)$ which vanish at infinity, i.e., for every $\varepsilon > 0$, there is $K \in \mathcal{K}$ such that $\sup_{t \in X \setminus K} w(t) \leq \varepsilon$.

In view of [14, Theorem 2.4] $\tau_c \subset \beta \subset \tau_u$ on $B(\mathcal{B}o)$ and β and τ_c coincide on any τ_u -bounded set in $B(\mathcal{B}o)$, where τ_c denotes the compact-open topology on $B(\mathcal{B}o)$. The topologies β and τ_u have the same bounded sets. If, in particular, X is compact, then $\beta = \tau_u$.

The following result characterizes a local base at 0 for β .

Proposition 1.1. *The strict topology β on $B(\mathcal{B}o)$ has a local base at 0 consisting of all sets of the form:*

$$\bigcap_{n=1}^{\infty} \left\{ v \in B(\mathcal{B}o) : a_n \sup_{t \in K_n} |v(t)| \leq 1 \right\},$$

where (K_n) is a sequence of compact sets in X and (a_n) is a sequence of positive numbers with $\lim a_n = 0$.

Proof. Assume that $w_o \in B_o(X)^+$ with $\|w_o\|_\infty = 1$. For $\varepsilon > 0$ let $V_{w_o}(\varepsilon) = \{v \in B(\mathcal{B}o) : p_{w_o}(v) \leq \varepsilon\}$. Choose a sequence (K_n) in \mathcal{K} such that $\sup_{t \in X \setminus K_n} w_o(t) \leq \frac{1}{n+1}$ for $n \in \mathbb{N}$. We can assume that $\emptyset = K_o \subset K_1 \subset \dots \subset K_n \subset \dots$. We shall show that

$$\bigcap_{n=1}^{\infty} \left\{ v \in B(\mathcal{B}o) : \frac{1}{n\varepsilon} \sup_{t \in K_n} |v(t)| \leq 1 \right\} \subset V_{w_o}(\varepsilon).$$

Indeed, assume that $v \in B(\mathcal{B}o)$ and $\frac{1}{n\varepsilon} \sup_{t \in K_n} |v(t)| \leq 1$ for $n \in \mathbb{N}$. Then for $t \in K_n$, $w_o(t)|v(t)| \leq \max_{1 \leq i \leq n} (\sup_{t \in K_i \setminus K_{i-1}} w_o(t)|v(t)|) \leq \varepsilon$. Since $w_o(t) = 0$ for $t \in X \setminus \bigcup_{n=1}^{\infty} K_n$, we have that $p_{w_o}(v) \leq \varepsilon$.

Conversely, let (K_n) be a sequence in \mathcal{K} and (a_n) be a sequence of positive numbers with $\lim a_n = 0$. Without loss of generality we can assume that $\emptyset = K_0 \subset K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$ and $a_n \downarrow 0$. Let $w_o = \sum_{n=1}^{\infty} a_n \mathbb{1}_{K_n \setminus K_{n-1}}$. Then $w_o \in B_o(X)^+$ and assume that $v \in V_{w_o}(1)$, i.e., $p_{w_o}(v) \leq 1$. Hence for

$t \in K_n$, $a_n|v(t)| \leq w_o(t)|v(t)| \leq 1$, i.e., $a_n \sup_{t \in K_n} |v(t)| \leq 1$. This means that $V_{w_o}(1) \subset \bigcap_{n=1}^{\infty} \{v \in B(\mathcal{B}o) : a_n \sup_{t \in K_n} |v(t)| \leq 1\}$. \square

The strict topology β restricted to $C_b(X)$ (denoted by β again) has been studied intensively (see [6, 7, 14, 17, 33]). Then β can be characterized as the finest locally convex Hausdorff topology on $C_b(X)$ which coincides with τ_c on τ_u -bounded sets (see [33, Theorem 2.4], [32]). This means that $(C_b(X), \beta)$ is a generalized DF-space (see [32, Corollary]); equivalently, β coincides with the mixed topology $\gamma[\tau_u, \tau_c]$ in the sense of Wiweger (see [38, (D), pp. 65–66], [7] for more details). If, in particular, X is locally compact, then β coincides with the original strict topology of Buck [4].

If X is compact, we will write simply $C(X)$ instead of $C_b(X)$. For a locally compact Hausdorff space X , $C_o(X)$ stands for the Banach space of all those functions of $C_b(X)$ that vanish at infinity in X .

The first studies of operators on spaces of scalar continuous functions were made independently in the fundamental papers of Grothendieck [15] and Bartle, Schwartz and Dunford [2]. In 1953, Grothendieck has showed that there is a one-to-one correspondence between the weakly compact operators from $C(X)$ into a complete lchS and lchS-valued Baire measures on a compact Hausdorff space X . But in [15] any theory of integration to represent these operators is not developed. In 1955, Bartle, Dunford and Schwartz [2] developed a theory of integration for scalar functions with respect to Banach space-valued measures and use it to give an integral representation for weakly compact operators $T : C(X) \rightarrow E$, where X is a compact Hausdorff space and E is a Banach space. Later, in 1970 Lewis [21] studied a Pettis type weak integral of scalar functions with respect to a countably additive lchS-valued measure. In particular, in [21] a Bartle-Dunford-Schwartz type theorem for weakly compact operators from $C(X)$ to a lchS is proved. Moreover, it is showed that if X is a locally compact Hausdorff space, then the space $C_b(X)$, equipped with the strict topology β has the Dunford-Pettis property. The study of continuous linear operators from the Banach space $C_o(X)$ to a lchS has been intensively developed by Edwards [12] and Panchapagesan in a series of papers [26–29] and a monograph [30].

When X is a completely regular Hausdorff space, continuous linear operator from $C_b(X)$, equipped with the different kinds of strict topologies β_z ($z = \sigma, \tau, t, p, g, s$), to a lchS (in particular, a Banach space) have been studied by Lewis [21], Khurana [19], Aguayo and Sanchez [1], Chacón and Vielma [5] and the present author [23, 24].

The aim of this paper is to build a general Riesz representation theory for (β, ξ) -continuous linear operators from $C_b(X)$ to an arbitrary lchS (E, ξ) , extending a number of results which are generally known to be true if X is a compact Hausdorff space and E is a Banach space. In Sect. 2, making use of the results of Topsoe [36] we state characterizations of weak compactness of bounded sets in the Banach space $M(X)$ of scalar Radon measures in case X is a k -space. These characterizations play a key role in the studies of operators

on $C_b(X)$. Further, we use it to obtain a generalization of a well-known result of Dieudonné for the weak sequential convergence in $M(X)$. In Sect. 3 we study the problem of an integral representation for (β, ξ) -continuous operators from $C_b(X)$ to (E, ξ) with respect to the representing Borel measures with values in the bidual E''_ξ of (E, ξ) . The strong integrability of functions in $B(\mathcal{B}o)$ is considered with respect to the completion $(\widetilde{E''_\xi}, \widetilde{\xi_\xi})$ of (E''_ξ, ξ_ξ) , where ξ_ξ stands for the natural topology on E''_ξ . It is shown that every (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$ possesses a (β, ξ_ξ) -continuous extension $\hat{T} : B(\mathcal{B}o) \rightarrow E''_\xi$. In Sect. 4, if X is a k -space and (E, ξ) is quasicomplete, we present equivalent conditions for a (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$ to be weakly compact, in particular, in terms of the representing measures. As a consequence, we derive that if X is a k -space and (E, ξ) is quasicomplete and contains no isomorphic copy of c_0 , then every (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$ is weakly compact.

2. Topological Properties of Spaces of Scalar Measures

Recall that a countably additive scalar measure μ on $\mathcal{B}o$ is called a *Radon measure* if its variation $|\mu|$ is regular, i.e., for each $A \in \mathcal{B}o$,

$$|\mu|(A) = \sup \{ |\mu|(K) : K \in \mathcal{K}, K \subset A \} = \inf \{ |\mu|(O) : O \in \mathcal{T}, O \supset A \}.$$

Let $M(X)$ denote the space of all Radon measures, equipped with the total variation norm $\|\mu\| := |\mu|(X)$. Note that for $\mu \in M(X)$ and $A \in \mathcal{B}o$ (see [10, Proposition 11, pp. 4–5]):

$$(2.1) \quad \sup \{ |\mu(B)| : B \subset A, B \in \mathcal{B}o \} \leq |\mu|(A) \leq 4 \sup \{ |\mu(B)| : B \subset A, B \in \mathcal{B}o \} < \infty.$$

The following characterization of the topological dual of $(C_b(X), \beta)$ will be of importance (see [14, Lemma 4.5]).

Theorem 2.1. *For a linear functional Φ on $C_b(X)$ the following statements are equivalent:*

- (i) Φ is β -continuous.
- (ii) There exists a unique $\mu \in M(X)$ such that $\Phi(u) = \Phi_\mu(u) = \int_X u d\mu$ for $u \in C_b(X)$.

It is known that for a sequence (u_n) in $C_b(X)$, $u_n \rightarrow 0$ in $\sigma(C_b(X), C_b(X)'_\beta)$ if and only if $\sup_n \|u_n\|_\infty < \infty$ and $u_n(t) \rightarrow 0$ for every $t \in X$ (see [20, Corollary 5]).

The following result will be useful (see [25, Lemma 2.5]).

Lemma 2.2. *Assume that $\mu \in M(X)$. Then for $A \in \mathcal{B}o$,*

$$|\mu|(A) = \sup \left\{ \left| \int_A u d\mu \right| : u \in C_b(X), \|u\|_\infty = 1 \right\}.$$

In particular, for $O \in \mathcal{T}$, we have

$$|\mu|(O) = \sup \left\{ \left| \int_O u \, d\mu \right| : u \in C_b(X), \|u\|_\infty = 1 \text{ and } \text{supp } u \subset O \right\}$$

and $\|\Phi_\mu\| = |\mu|(X)$.

It is known that $C_b(X)'_\beta$ is equal to the closure of $C_b(X)'_{\tau_c}$ in the Banach space $(C_b(X)', \|\cdot\|')$ (see [7, Proposition 1]). Hence in view of Lemma 2.2, the space $M(X)$ (equipped with the total variation norm $\|\mu\| = |\mu|(X)$) is a Banach space.

We will need the following result (see [33, Theorem 5.1]).

Theorem 2.3. *For a subset \mathcal{M} of $M(X)$ the following statements are equivalent:*

- (i) $\{\Phi_\mu : \mu \in \mathcal{M}\}$ is β -equicontinuous.
- (ii) $\sup_{\mu \in \mathcal{M}} |\mu|(X) < \infty$ and $\{|\mu| : \mu \in \mathcal{M}\}$ is uniformly tight, i.e., for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that $\sup_{\mu \in \mathcal{M}} |\mu|(X \setminus K) \leq \varepsilon$.

Recall that a completely regular Hausdorff space (X, \mathcal{T}) is a k -space if a subset A of X is closed whenever $A \cap K$ is compact for all compact sets in X . In particular, every locally compact Hausdorff space, every metrizable space and every space satisfying the first countability axiom is a k -space (see [13, Chap. 3, § 3]).

Now using the results of Topsoe's paper [36] we can state the following extension to k -spaces of the celebrated Dieudonné-Grothendieck's criterion on relative weak compactness in the space $M(X)$ (see [15, Theorem 2], [9, Theorem 14, pp. 98–103]), which plays a crucial role in the study of operators on $C_b(X)$.

By τ_s we denote the topology of simple convergence in $ca(\mathcal{B}o)$. Then τ_s is generated by the family $\{p_A : A \in \mathcal{B}o\}$ of seminorms, where $p_A(\mu) = |\mu(A)|$ for $\mu \in ca(\mathcal{B}o)$.

Theorem 2.4. *Assume that X is a k -space and \mathcal{M} is a subset of $M(X)$ such that $\sup_{\mu \in \mathcal{M}} |\mu|(X) < \infty$. Then the following statements are equivalent:*

- (i) \mathcal{M} is relatively weakly compact in the Banach space $M(X)$.
- (ii) \mathcal{M} is uniformly countably additive, i.e., $\sup_{\mu \in \mathcal{M}} |\mu(A_n)| \rightarrow 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathcal{B}o$.
- (iii) $\sup_{\mu \in \mathcal{M}} |\mu|(O_n) \rightarrow 0$ whenever (O_n) is a pairwise disjoint sequence of open sets.
- (iv) $\sup_{\mu \in \mathcal{M}} |\mu(O_n)| \rightarrow 0$ whenever (O_n) is a pairwise disjoint sequence of open sets.
- (v) $\{|\mu| : \mu \in \mathcal{M}\}$ is uniformly regular, i.e., for every $A \in \mathcal{B}o$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ with $K \subset A \subset O$ such that $\sup_{\mu \in \mathcal{M}} |\mu|(O \setminus K) \leq \varepsilon$.
- (vi) \mathcal{M} is uniformly regular, i.e., for every $A \in \mathcal{B}o$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ with $K \subset A \subset O$ such that $\sup_{\mu \in \mathcal{M}} |\mu(B)| \leq \varepsilon$ for every $B \in \mathcal{B}o$ with $B \subset O \setminus K$.

- (vii) $\sup_{\mu \in \mathcal{M}} |\int_X u_n d\mu| \rightarrow 0$ whenever (u_n) is a uniformly bounded sequence in $C_b(X)$ such that $u_n(t) \rightarrow 0$ for every $t \in X$.
- (viii) $\sup_{\mu \in \mathcal{M}} |\int_X u_n d\mu| \rightarrow 0$ whenever $u_n \rightarrow 0$ in $\sigma(C_b(X), C_b(X)'_\beta)$.
- (ix) $\sup_{\mu \in \mathcal{M}} |\int_X u_n d\mu| \rightarrow 0$ whenever (u_n) is a uniformly bounded sequence in $C_b(X)$ such that $\text{supp } u_n \cap \text{supp } u_k = \emptyset$ for $n \neq k$.

Proof. (i) \Leftrightarrow (ii) It follows from [9, Chap. 7, Theorem 13] because $M(X)$ is a closed subspace of the Banach space $ca(\mathcal{B}o)$, equipped with the norm $\|\mu\| = |\mu|(X)$ (see [18, Chap. 3, § 3, Corollary 3]).

(ii) \Rightarrow (iii) Assume that (ii) holds. Then $\sup_{\mu \in \mathcal{M}} |\mu|(A_n) \rightarrow 0$ for every pairwise disjoint sequence (A_n) in $\mathcal{B}o$ (see [9, Theorem 10, pp. 88–89], [10, Proposition 17, p. 8]). Hence (iii) holds.

(iii) \Rightarrow (iv) It is obvious.

(iv) \Rightarrow (iii) Assume that (iii) does not hold. Then there exist $\varepsilon > 0$, a sequence (O_n) of pairwise disjoint sets in \mathcal{T} and a sequence (μ_n) in \mathcal{M} such that $|\mu_n|(O_n) \geq \varepsilon$ for $n \in \mathbb{N}$. By (2.1) for every $n \in \mathbb{N}$ there exists $A_n \in \mathcal{B}o$ with $A_n \subset O_n$ such that $|\mu_n|(O_n) \leq 4|\mu_n(A_n)| + \frac{\varepsilon}{4}$. By the regularity of μ_n , there exists $O'_n \in \mathcal{T}$ with $A_n \subset O'_n$ such that $|\mu_n|(O'_n \setminus A_n) \leq \frac{\varepsilon}{8}$. Let $U_n = O_n \cap O'_n$ for $n \in \mathbb{N}$. Then (U_n) is a pairwise disjoint sequence in \mathcal{T} and $|\mu_n(A_n)| \leq |\mu_n(U_n)| + |\mu_n(U_n \setminus A_n)|$ for $n \in \mathbb{N}$. Hence $|\mu_n(U_n \setminus A_n)| \leq |\mu_n|(U_n \setminus A_n) \leq |\mu_n|(O'_n \setminus A_n) \leq \frac{\varepsilon}{8}$ for $n \in \mathbb{N}$, and we get $\varepsilon \leq |\mu_n|(O_n) \leq 4|\mu_n(U_n)| + \frac{3}{4}\varepsilon$ and hence $|\mu_n(U_n)| \geq \frac{1}{16}\varepsilon$. It follows that (iv) does not hold.

(iii) \Rightarrow (v) Assume that (iii) holds. Then by [T, Theorem 8] the set $\{|\mu| : \mu \in \mathcal{M}\}$ is relatively $\tau_s|_{M^+(X)}$ -compact in $M^+(X)$, where $(M^+(X) := \{\mu \in M(X) : \mu(A) \geq 0 \text{ for every } A \in \mathcal{B}o\})$. Hence by [35, Proposition 1] (v) holds.

(v) \Leftrightarrow (vi) It follows from (2.1).

(v) \Rightarrow (ii) It is enough to repeat verbatim the proof of implication (d) \Rightarrow (e) in [10, Lemma 13, pp. 157–159].

(ii) \Rightarrow (vii) Assume that \mathcal{M} is uniformly countably additive. Let (u_n) be a sequence in $C_b(X)$ such that $\sup_n \|u_n\|_\infty = a < \infty$ and $u_n(t) \rightarrow 0$ for every $t \in X$. Then there exists $\lambda \in ca(\mathcal{B}o)^+$ such that $\lim_{\lambda(A) \rightarrow 0} \sup_{\mu \in \mathcal{M}} |\mu|(A) = 0$. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$ such that $\sup_{\mu \in \mathcal{M}} |\mu|(A) \leq \frac{\varepsilon}{2(a+1)}$ whenever $\lambda(A) \leq \eta$, $A \in \mathcal{B}o$. Hence by the Egoroff theorem there exists $A_\eta \in \mathcal{B}o$ with $\lambda(X \setminus A_\eta) \leq \eta$ and $\sup_{t \in A_\eta} |u_n(t)| \rightarrow 0$, so $\sup_{t \in A_\eta} |u_n(t)| \leq \frac{\varepsilon}{2(c+1)}$ for $n \geq n_o$ for some $n_o \in \mathbb{N}$, where $c = \sup_{\mu \in \mathcal{M}} |\mu|(X)$. Then for every $\mu \in \mathcal{M}$ and $n \geq n_o$, we have

$$\begin{aligned} \left| \int_X u_n d\mu \right| &\leq \int_X |u_n| d|\mu| = \int_{A_\eta} |u_n| d|\mu| + \int_{X \setminus A_\eta} |u_n| d|\mu| \\ &\leq \frac{\varepsilon}{2(a+1)} |\mu|(A_\eta) + a |\mu|(X \setminus A_\eta) \leq \frac{\varepsilon}{2(c+1)} c + a \frac{\varepsilon}{2(a+1)} \leq \varepsilon. \end{aligned}$$

It follows that $\sup_{\mu \in \mathcal{M}} |\int_X u_n d\mu| \leq \varepsilon$ for $n \geq n_o$.

(vii) \Leftrightarrow (viii) It is obvious.

(vii) \Rightarrow (ix) It is obvious.

(ix) \Rightarrow (iv) Assume that (iv) does not hold. Hence there exist $\varepsilon_o > 0$, a pairwise disjoint sequence (O_n) in \mathcal{T} and a sequence (μ_n) in \mathcal{M} such that $|\mu_n(O_n)| > \varepsilon_o$. In view of Lemma 2.2 one can choose a sequence (u_n) in $C_b(X)$ with $\|u_n\|_\infty = 1$ and $\text{supp } u_n \subset O_n$ such that

$$\left| \int_{O_n} u_n d\mu_n \right| \geq |\mu_n|(O_n) - \frac{\varepsilon_o}{2} > \frac{\varepsilon_o}{2}.$$

Then for $n \in \mathbb{N}$,

$$\sup_{\mu \in \mathcal{M}} \left| \int_X u_n d\mu \right| \geq \left| \int_{O_n} u_n d\mu_n \right| > \frac{\varepsilon_o}{2}.$$

Since $\text{supp } u_n \cap \text{supp } u_k = \emptyset$ for $n \neq k$, we derive that the condition (ix) does not hold. \square

Remark 2.1. The problem of weak compactness of bounded sets of $M(X)$ if X is a locally compact Hausdorff space has been studied by Edwards [12, Theorem 4.22.1] and Panchapagesan [28, Theorems 1 and 2]. In case X is a compact Hausdorff space, an analogous result to Theorem 2.4 can be found in [10, Lemma 13, pp. 157–159] and in [9, Theorem 14, pp. 98–103].

In 1951 Dieudonné [8] proved that, if a sequence of Radon measures defined on the Borel σ -algebra of a compact metrizable space converges on every open set, then it converges on every Borel set, and in this case the sequence is uniformly regular. Brooks [3] generalizes this theorem to the case the space is either compact or the space is normal and the sequence is uniformly bounded.

As an application of Theorem 2.4 we can state a Dieudonné-type theorem in the setting of k -spaces.

Corollary 2.5. *Assume that X is a k -space and (μ_n) is a bounded sequence in the Banach space $M(X)$. Then the following statements are equivalent:*

- (i) $\mu_n \rightarrow \mu$ in $\sigma(M(X), M(X)')$ for some $\mu \in M(X)$.
- (ii) For every $A \in \mathcal{B}_o$, $\lim \mu_n(A)$ exists and the set $\{\mu_n : n \in \mathbb{N}\}$ is uniformly regular.
- (iii) For every open set O in X , $\lim \mu_n(O)$ exists.

Proof. (i) \Rightarrow (ii) Assume that (i) holds. Clearly $\mu_n(A) \rightarrow \mu(A)$ for every $A \in \mathcal{B}_o$. Since the set $\{\mu_n : n \in \mathbb{N}\} \cup \{\mu\}$ is $\sigma(M(X), M(X)')$ -compact, by Theorem 2.4 we obtain that the set $\{\mu_n : n \in \mathbb{N}\}$ is uniformly regular.

(ii) \Rightarrow (iii) It is obvious.

(iii) \Rightarrow (i) Assume that (iii) holds. Then using Theorem 2.4 and arguing as in the proof of Corollary 4.22.2 of [12] we obtain that (i) holds. \square

Remark 2.2. An analogous result to Corollary 2.5 for X a locally compact Hausdorff space can be found in [12, Corollary 4.22.2] and [29, Corollary 1].

As a consequence of Corollary 2.5 we have:

Corollary 2.6. *Assume that X is a k -space. Then the Banach space $M(X)$ is weakly sequentially complete.*

3. Integral Representation of Operators on $C_b(X)$

Let $C_b(X)''_\beta$ denote the bidual of $(C_b(X), \beta)$. Since β -bounded subsets of $C_b(X)$ are τ_u -bounded, the strong topology $\beta(C_b(X)'_\beta, C_b(X))$ in $C_b(X)'_\beta$ coincides with the $\|\cdot\|'$ -norm topology in $C_b(X)'$ restricted to $C_b(X)'_\beta$. Hence $C_b(X)''_\beta = (C_b(X)'_\beta, \|\cdot\|')'$ and $\|\Psi\|'' = \sup\{|\Psi(\Phi)| : \Phi \in C_b(X)'_\beta, \|\Phi\|' \leq 1\}$ for $\Psi \in C_b(X)''_\beta$. Then one can embed $B(\mathcal{B}o)$ into $C_b(X)''_\beta$ by the mapping $\pi : B(\mathcal{B}o) \rightarrow C_b(X)''_\beta$, where for $v \in B(\mathcal{B}o)$,

$$\pi(v)(\Phi_\mu) := \int_X v d\mu \quad \text{for } \mu \in M(X).$$

We have

$$|\pi(v)(\Phi_\mu)| = \left| \int_X v d\mu \right| \leq \|v\|_\infty |\mu|(X) = \|v\|_\infty \|\Phi_\mu\|'$$

and hence π is bounded and $\|\pi(v)\|'' \leq \|v\|_\infty$.

For $t \in X$ let $\delta_t(u) := u(t)$ for $u \in C_b(X)$. Then $\delta_t \in C_b(X)'_\beta$ with $\|\delta_t\|' = 1$, and there exists $\mu_t \in M(X)$ such that $\delta_t(u) = \int_X u d\mu_t$ for $u \in C_b(X)$. Moreover, μ_t is concentrated on $\{t\}$, that is, $\mu_t(A) = \mathbb{1}_A(t)$ for $A \in \mathcal{B}o$. Hence, we get

$$\begin{aligned} \|\pi(v)\|'' &\geq \sup\{|\pi(v)(\delta_t)| : t \in X\} = \sup\left\{\left|\int_X v d\mu_t\right| : t \in X\right\} \\ &= \sup\{|v(t)| : t \in X\} = \|v\|_\infty. \end{aligned}$$

It follows that $\|\pi(v)\|'' = \|v\|_\infty$.

Let E''_ξ be the bidual of (E, ξ) , i.e., $E''_\xi = (E'_\xi, \beta(E'_\xi, E))'$. Let \mathcal{E} denote the family of all ξ -equicontinuous subsets of E'_ξ . Then ξ is generated by the family of seminorms $\{p_D : D \in \mathcal{E}\}$, where

$$p_D(e) := \sup\{|e'(e)| : e' \in D\} \quad \text{for } e \in E.$$

The so-called *natural topology* $\xi_\mathcal{E}$ on E''_ξ is generated by the family of seminorms $\{q_D : D \in \mathcal{E}\}$, where

$$q_D(e'') := \sup\{|e''(e')| : e' \in D\} \quad \text{for } e'' \in E''_\xi$$

(see [12, § 8.7] for more details).

Let $i_E : E \rightarrow E''_\xi$ stand for the canonical injection, that is, $i_E(e)(e') = e'(e)$ for $e \in E$ and $e' \in E'_\xi$. Let $j_E : i_E(E) \rightarrow E$ stand for the left inverse of i_E , i.e., $j_E(i_E(e)) = e$ for $e \in E$. Then $i_E : E \rightarrow i_E(E)$ is a $(\xi, \xi_\mathcal{E}|_{i_E(E)})$ -homeomorphism.

Assume that $T : C_b(X) \rightarrow E$ is a (β, ξ) -continuous linear operator. Then T is $(\sigma(C_b(X), C_b(X)'_\beta), \sigma(E, E'_\xi))$ -continuous (see [12, Corollary 8.6.5]) and one can define the conjugate mapping

$$T' : E'_\xi \rightarrow C_b(X)'_\beta$$

by putting $T'(e') := e' \circ T$ for $e' \in E'_\xi$. Then T' is $(\beta(E'_\xi, E), \beta(C_b(X)'_\beta, C_b(X)))$ -continuous (see [12, Proposition 8.7.1]) and hence T' is $(\sigma(E'_\xi, E), \sigma(C_b(X)'_\beta, C_b(X)))$ -continuous. It follows that we can define the biconjugate mapping

$$T'' : C_b(X)''_\beta \rightarrow E''_\xi$$

by putting $T''(\Psi)(e') = \Psi(T'(e'))$ for $\Psi \in C_b(X)''_\beta$ and $e' \in E'_\xi$. Then T'' is $(\sigma(C_b(X)''_\beta, C_b(X)'_\beta), \sigma(E''_\xi, E'_\xi))$ -continuous. Since the topology $(\tau_u)_\mathcal{E}$ on $C_b(X)''_\beta$ coincides with the $\|\cdot\|$ -norm topology on $C_b(X)''$ restricted to $C_b(X)''_\beta$, in view of [12, Proposition 8.7.2] T'' is $(\|\cdot\|_{C_b(X)''_\beta}, \xi_\mathcal{E})$ -continuous. Let

$$\hat{T} := T'' \circ \pi : B(\mathcal{B}o) \rightarrow E''_\xi.$$

Then \hat{T} is a $(\tau_u, \xi_\mathcal{E})$ -continuous linear operator. For $A \in \mathcal{B}o$ let

$$\hat{m}(A) := \hat{T}(\mathbb{1}_A).$$

Hence $\hat{m} : \mathcal{B}o \rightarrow E''_\xi$ is a finitely additive measure with the $\xi_\mathcal{E}$ -bounded range and is called a *representing measure* of T . For every $e' \in E'_\xi$, let

$$\hat{m}_{e'}(A) := \hat{m}(A)(e') \quad \text{for } A \in \mathcal{B}o.$$

From the general properties of the operator \hat{T} it follows immediately that

$$\hat{T}(C_b(X)) \subset i_E(E) \quad \text{and} \quad T(u) = j_E(\hat{T}(u)) \quad \text{for } u \in C_b(X).$$

From now on we will use the integration theory of scalar functions with respect to vector measures that is developed in [16, 26, 30].

In view of (2.1) for $D \in \mathcal{E}$ and $A \in \mathcal{B}o$, we have

$$\begin{aligned} \sup \{q_D(\hat{m}(B)) : B \in \mathcal{B}o, B \subset A\} &\leq \sup_{e' \in D} |\hat{m}_{e'}|(A) \\ &\leq 4 \sup \{q_D(\hat{m}(B)) : B \in \mathcal{B}o, B \subset A\} < \infty. \end{aligned} \quad (3.1)$$

It follows that $\sup_{e' \in D} |\hat{m}_{e'}|(X) < \infty$ for every $D \in \mathcal{E}$. Let $v \in B(\mathcal{B}o)$. Choose a sequence (s_n) in $\mathcal{S}(\mathcal{B}o)$ such that $\|s_n - v\|_\infty \rightarrow 0$. Then $(\int_X s_n d\hat{m})$ is a $\xi_\mathcal{E}$ -Cauchy sequence in E''_ξ because for every $D \in \mathcal{E}$ and $k, n \in \mathbb{N}$, we have

$$q_D \left(\int_X s_n d\hat{m} - \int_X s_k d\hat{m} \right) \leq \|s_n - s_k\|_\infty \cdot \sup_{e' \in D} |\hat{m}_{e'}|(X). \quad (3.2)$$

Then every $v \in B(\mathcal{B}o)$ is \hat{m} -integrable with respect to the completion $(\widetilde{E''_\xi}, \widetilde{\xi_\mathcal{E}})$ of $(E''_\xi, \xi_\mathcal{E})$, and one can define the integral $\int_X v d\hat{m}$ by

$$\int_X v d\hat{m} := \widetilde{\xi_\mathcal{E}} - \lim_n \int_X s_n d\hat{m} \quad \text{in } \widetilde{E''_\xi}. \quad (3.3)$$

Definition 3.1. A finitely additive measure $\hat{m} : \mathcal{Bo} \rightarrow E''_\xi$ is said to be $\xi_\mathcal{E}$ -tight if for every $D \in \mathcal{E}$ and $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that $q_D(\hat{m}(B)) = \sup_{e' \in D} |\hat{m}_{e'}(B)| \leq \varepsilon$ for every $B \in \mathcal{Bo}$ with $B \subset X \setminus K$ (equivalently; $\sup_{e' \in D} |\hat{m}_{e'}|(X \setminus K) \leq \varepsilon$).

Now we can state a general Riesz representation theorem for continuous linear operators from $(C_b(X), \beta)$ to a lchS (E, ξ) .

Theorem 3.1. Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous linear operator and \hat{m} be its representing measure. Then the following statements hold:

- (i) For every $e' \in E'_\xi$, $\hat{m}_{e'} \in M(X)$.
- (ii) For every $e' \in E'_\xi$, $e'(T(u)) = \int_X u d\hat{m}_{e'}$ for $u \in C_b(X)$.
- (iii) The mapping $E'_\xi \ni e' \mapsto \hat{m}_{e'} \in M(X)$ is $(\sigma(E'_\xi, E), \sigma(M(X), C_b(X)))$ -continuous.
- (iv) \hat{m} is $\xi_\mathcal{E}$ -bounded (i.e., $\sup_{e' \in D} |\hat{m}_{e'}|(X) < \infty$ for every $D \in \mathcal{E}$) and $\xi_\mathcal{E}$ -tight.
- (v) For every $v \in B(\mathcal{Bo})$, we have

$$\hat{T}(v) = \int_X v d\hat{m} \quad \text{and} \quad \hat{T}(v)(e') = \int_X v d\hat{m}_{e'} \quad \text{for every } e' \in E'_\xi.$$

- (vi) $\hat{T} : B(\mathcal{Bo}) \rightarrow E''_\xi$ is $(\beta, \xi_\mathcal{E})$ -continuous.

Conversely, let $\hat{m} : \mathcal{Bo} \rightarrow E''_\xi$ be a finitely additive measure satisfying (i), (iii) and (iv). Then there exists a unique (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$ such that for every $e' \in E'_\xi$, $e'(T(u)) = \int_X u d\hat{m}_{e'}$ for $u \in C_b(X)$. Moreover, \hat{m} is the representing measure of T .

Proof. (i) Let $e' \in E'_\xi$. Since $e' \circ T \in C_b(X)'_\beta$, by Theorem 2.1 there exists a unique $\mu_{e'} \in M(X)$ such that $(e' \circ T)(u) = \int_X u d\mu_{e'} = \Phi_{\mu_{e'}}(u)$ for $u \in C_b(X)$. For $A \in \mathcal{Bo}$, we have

$$\hat{m}_{e'}(A) = T''(\pi(\mathbb{1}_A))(e') = \pi(\mathbb{1}_A)(T'(e')) = \pi(\mathbb{1}_A)(e' \circ T) = \int_X \mathbb{1}_A d\mu_{e'} = \mu_{e'}(A).$$

It follows that $\hat{m}_{e'} = \mu_{e'} \in M(X)$.

- (ii) In view of (i) for every $e' \in E'_\xi$, $e'(T(u)) = \int_X u d\hat{m}_{e'}$ for $u \in C_b(X)$.
- (iii) Since the mapping $T' : E'_\xi \rightarrow C_b(X)'_\beta$ is $(\sigma(E'_\xi, E), \sigma(C_b(X)'_\beta, C_b(X)))$ -continuous, the mapping $E'_\xi \ni e' \mapsto \hat{m}_{e'} \in M(X)$ is $(\sigma(E'_\xi, E), \sigma(M(X), C_b(X)))$ -continuous.

(iv) It follows from (ii) and Theorem 2.3 because for every $D \in \mathcal{E}$, the family $\{e' \circ T : e' \in D\}$ is β -equicontinuous.

(v) Let $v \in B(\mathcal{Bo})$. Choose a sequence (s_n) in $\mathcal{S}(\mathcal{Bo})$ such that $\|v - s_n\|_\infty \rightarrow 0$. Since $\hat{T} : B(\mathcal{Bo}) \rightarrow E''_\xi$ is $(\tau_u, \xi_\mathcal{E})$ -continuous, we get

$$\hat{T}(v) = \xi_\mathcal{E} - \lim_n \hat{T}(s_n) = \xi_\mathcal{E} - \lim \int_X s_n d\hat{m}, \quad (3.4)$$

where $\int_X s_n d\hat{m} = \hat{T}(s_n) \in E''_\xi$. Hence using (3.3) we have $\hat{T}(v) = \int_X v d\hat{m}$.

Let $e' \in E'_\xi$. Define a linear functional $\varphi_{e'}$ on $B(\mathcal{B}o)$ by $\varphi_{e'}(v) = \hat{T}(v)(e')$ for $v \in B(\mathcal{B}o)$. Note that $\varphi_{e'} = i_{e'} \circ \hat{T}$, where $i_{e'}$ is a linear functional on E''_ξ defined by $i_{e'}(e'') = e''(e')$ for $e'' \in E''_\xi$. Since $i_{e'}$ is $\xi_\mathcal{E}$ -continuous, we obtain that $\varphi_{e'}$ is τ_u -continuous. Hence there exists a unique $\mu_{e'} \in ba(\mathcal{B}o)$ such that $\varphi_{e'}(v) = \int_X v d\mu_{e'}$ for $v \in B(\mathcal{B}o)$ (see [9, Theorem 7, p. 77]). Then for $A \in \mathcal{B}o$, $\hat{m}_{e'}(A) = \hat{T}(\mathbb{1}_A)(e') = \varphi_{e'}(\mathbb{1}_A) = \mu_{e'}(A)$, so $\hat{m}_{e'} = \mu_{e'}$ and $\hat{T}(v)(e') = \varphi_{e'}(v) = \int_X v d\hat{m}_{e'}$ for $v \in B(\mathcal{B}o)$.

(vi) Let $D \in \mathcal{E}$ and $\varepsilon > 0$ be given. By (iv) there exists a sequence (K_n) in \mathcal{K} with $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ such that $\sup_{e' \in D} |\hat{m}_{e'}|(X \setminus K_n) \leq 2^{-2n}$ for $n \in \mathbb{N}$.

Let $c_D = \max(1, \sup_{e' \in D} |\hat{m}_{e'}|(X))$ and $\eta = \frac{\varepsilon}{c_D}$. Then by Proposition 1.1 the set

$$U(D, \varepsilon) = \bigcap_{n=1}^{\infty} \left\{ v \in B(\mathcal{B}o) : \eta^{-1} 2^{-n+2} \sup_{t \in K_n} |v(t)| \leq 1 \right\}$$

is a β -neighborhood of 0 in $B(\mathcal{B}o)$. Note that for $e' \in D$, $|\hat{m}_{e'}|(X \setminus \bigcup_{n=1}^{\infty} K_n) = 0$. Hence for every $v \in U(D, \varepsilon)$ and $e' \in D$, we get

$$\begin{aligned} |\hat{T}(v)(e')| &= \left| \int_X v d\hat{m}_{e'} \right| = \left| \int_{\bigcup_{n=1}^{\infty} K_n} v d\hat{m}_{e'} + \int_{X \setminus \bigcup_{n=1}^{\infty} K_n} v d\hat{m}_{e'} \right| \\ &\leq \int_{K_1} |v| d|\hat{m}_{e'}| + \int_{\bigcup_{n=1}^{\infty} (K_{n+1} \setminus K_n)} |v| d|\hat{m}_{e'}| + \int_{X \setminus \bigcup_{n=1}^{\infty} K_n} |v| d|\hat{m}_{e'}| \\ &\leq \sup_{t \in K_1} |v(t)| \cdot |\hat{m}_{e'}|(X) + \sum_{n=1}^{\infty} \int_{K_{n+1} \setminus K_n} |v| d|\hat{m}_{e'}| \\ &\leq \eta 2^{-1} c_D + \sum_{n=1}^{\infty} \sup_{t \in K_{n+1}} |v(t)| \cdot |\hat{m}_{e'}|(K_{n+1} \setminus K_n) \leq \frac{1}{2} \varepsilon + \sum_{n=1}^{\infty} \eta 2^{n-1} \cdot 2^{-2n} \leq \varepsilon. \end{aligned}$$

It follows that $q_D(\hat{T}(v)) \leq \varepsilon$, and this means that $\hat{T} : B(\mathcal{B}o) \rightarrow E''_\xi$ is $(\beta, \xi_\mathcal{E})$ -continuous.

Conversely, let $\hat{m} : \mathcal{B}o \rightarrow E''_\xi$ be a finitely additive measure satisfying the conditions (i), (iii) and (iv). For $u \in C_b(X)$ define a linear functional ψ_u on E'_ξ by $\psi_u(e') = \int_X u d\hat{m}_{e'}$ for $e' \in E'_\xi$. Then by (iii) ψ_u is $(\sigma(E'_\xi, E))$ -continuous, so there is a unique $e_u \in E$ such that $\psi_u(e') = e'(e_u)$ for each $e' \in E'_\xi$. For each $u \in C_b(X)$ let us put $T(u) := e_u$. Then $T : C_b(X) \rightarrow E$ is a linear mapping and for every $e' \in E'_\xi$, we have

$$e'(T(u)) = e'(e_u) = \psi_u(e') = \int_X u d\hat{m}_{e'} \quad \text{for } u \in C_b(X).$$

In view of (i) and (iv) and Theorem 2.3 for every $D \in \mathcal{E}$, the family $\{e' \circ T : e' \in D\}$ is β -equicontinuous and it follows that T is (β, ξ) -continuous.

Assume that $S : C_b(X) \rightarrow E$ is another (β, ξ) -continuous linear operator such that for each $e' \in E'_\xi$, $e'(S(u)) = \int_X u d\hat{m}_{e'}$ for all $u \in C_b(X)$. Then $e'(S(u)) = e'(T(u))$ for all $u \in C_b(X)$, i.e., $S = T$.

Let \hat{m}_o be the representing measure of T . Then by the first part of the proof, for each $e' \in E'_\xi$, we get $e'(T(u)) = \int_X u d(\hat{m}_o)_{e'}$ for every $u \in C_b(X)$, where $(\hat{m}_o)_{e'} \in M(X)$. It follows that $(\hat{m}_o)_{e'} = (\hat{m})_{e'}$ for each $e' \in E'_\xi$, i.e., $\hat{m}(A)(e') = \hat{m}_o(A)(e')$ for every $A \in \mathcal{B}_o$. Hence $\hat{m}_o = \hat{m}$. \square

Remark 3.1. A Riesz representation theorem for bounded linear operators $T : C(X) \rightarrow E$, where X is a compact Hausdorff space and E is a Banach space was proved by Bartle, Dunford and Schwartz (see [2, Theorem 3.1], [10, Theorem 1, pp. 152–153]). An analogous theorem for (τ_u, ξ) -continuous linear operators $T : C_o(X) \rightarrow E$, where X is a locally compact Hausdorff space and (E, ξ) is a lchS was proved by Panchapagesan ([27, Theorem 1]).

Now we study (β, ξ) -continuous linear operators $T : C_b(X) \rightarrow E$ such that the set $T'(D)$ is relatively $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact for every $D \in \mathcal{E}$ (see [12, Corollary 9.3.2]).

Proposition 3.2. *Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous linear operator. Then the following statements are equivalent:*

- (i) *For every $D \in \mathcal{E}$, $T'(D)$ is relatively $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact.*
- (ii) *$T'' : C_b(X)''_\beta \rightarrow E''_\xi$ is $(\tau(C_b(X)''_\beta, C_b(X)'_\beta), \xi_\mathcal{E})$ -continuous.*

Proof. (i) \Rightarrow (ii) Assume that (i) holds and V is a $\xi_\mathcal{E}$ -neighborhood of 0 in E''_ξ . Then there exists $D \in \mathcal{E}$ such that $D^0 \subset V$, where D^0 denotes the polar of D with respect to the dual pair $\langle E'_\xi, E''_\xi \rangle$. Then $T'(D)$ is a relatively $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact and by the Krein-Smulian theorem, its closed absolutely convex hull C is still $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact. Hence C^0 is a $\tau(C_b(X)''_\beta, C_b(X)'_\beta)$ -neighborhood of 0 in $C_b(X)''_\beta$, where C^0 denotes the polar of C with respect to the dual pair $\langle C_b(X)'_\beta, C_b(X)''_\beta \rangle$. Then $T''(C^0) \subset D^0 \subset V$ (see [12, § 8.6, (6.6.3)]) and this means that T'' is $(\tau(C_b(X)''_\beta, C_b(X)'_\beta), \xi_\mathcal{E})$ -continuous.

(ii) \Rightarrow (i) Assume that (ii) holds and $D \in \mathcal{E}$. It follows that there exists an absolutely convex $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact set C in $C_b(X)'_\beta$ such that $T''(C^0) \subset D^0$. It follows that $T'(D) \subset C$ (see [12, § 8.6, (8.6.3)]) and hence $T'(D)$ is relatively $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact. \square

Corollary 3.3. *Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous linear operator. If for every $D \in \mathcal{E}$, $T'(D)$ is a relatively $\sigma(C_b(X)'_\beta, C_b(X)''_\beta)$ -compact subset of $C_b(X)'_\beta$, then the following statements hold:*

- (i) *For every $v \in B(\mathcal{B}_o)$ there exists a net (u_α) in $C_b(X)$ with $\|u_\alpha\|_\infty \leq \|v\|_\infty$ such that $\hat{T}(u_\alpha) \rightarrow \hat{T}(v)$ in $\xi_\mathcal{E}$.*
- (ii) *$i_E(T(C_b(X)))$ is $\xi_\mathcal{E}$ -dense in $\hat{T}(B(\mathcal{B}_o))$.*

Proof. (i) Let $v \in B(\mathcal{B}_o)$ and $U_v = \{u \in C_b(X) : \|u\|_\infty \leq \|v\|_\infty\}$. Then by the Mazur theorem and the Goldstein type theorem (see [31, § 4, Theorem 4,

p. 35]), we get

$$\begin{aligned} \text{cl}_{\tau(C_b(X)''_{\beta}, C_b(X)'_{\beta})}(\pi(U_v)) &= \text{cl}_{\sigma(C_b(X)''_{\beta}, C_b(X)'_{\beta})}(\pi(U_v)) \\ &= \{\Psi \in C_b(X)''_{\beta} : \|\Psi\|'' \leq \|v\|_{\infty}\}. \end{aligned}$$

Since $\|\pi(v)\|'' = \|v\|_{\infty}$, there exists a net (u_{α}) in $C_b(X)$ with $\|u_{\alpha}\|_{\infty} \leq \|v\|_{\infty}$ such that $\pi(u_{\alpha}) \rightarrow \pi(v)$ in $\tau(C_b(X)''_{\beta}, C_b(X)'_{\beta})$. Hence by Proposition 3.2, $\hat{T}(u_{\alpha}) = T''(\pi(u_{\alpha})) \rightarrow T''(\pi(v)) = \hat{T}(v)$ in $\xi_{\mathcal{E}}$.

(ii) It follows from (i). \square

Remark 3.2. If X is a compact Hausdorff space, a related result to Corollary 3.3 was obtained by Shuchat [34, Proposition 1].

Definition 3.2 (see [11, § 3]). A finitely additive measure $\hat{m} : \mathcal{B}_0 \rightarrow E''_{\xi}$ is said to be $\xi_{\mathcal{E}}$ -regular if for every $D \in \mathcal{E}$, $A \in \mathcal{B}_0$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ with $K \subset A \subset O$ such that $q_D(\hat{m}(B)) = \sup_{e' \in D} |\hat{m}_{e'}(B)| \leq \varepsilon$ for every $B \in \mathcal{B}_0$ with $B \subset O \setminus K$ (equivalently, $\sup_{e' \in D} |\hat{m}_{e'}|(O \setminus K) \leq \varepsilon$).

Now we present equivalent conditions for a (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$, the set $T'(D)$ to be relatively $\sigma(C_b(X)'_{\beta}, C_b(X)''_{\beta})$ -compact for every $D \in \mathcal{E}$.

Corollary 3.4. Assume that X is a k -space. Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous linear operator and \hat{m} be its representing measure. Then the following statements are equivalent:

- (i) For every $D \in \mathcal{E}$, $T'(D)$ is relatively $\sigma(C_b(X)'_{\beta}, C_b(X)''_{\beta})$ -compact.
- (ii) For every $D \in \mathcal{E}$, $\{\hat{m}_{e'} : e' \in D\}$ is relatively $\sigma(M(X), M(X)')$ -compact.
- (iii) $\hat{m} : \mathcal{B}_0 \rightarrow E''_{\xi}$ is $\xi_{\mathcal{E}}$ -countably additive.
- (iv) $\hat{m} : \mathcal{B}_0 \rightarrow E''_{\xi}$ is $\xi_{\mathcal{E}}$ -regular.
- (v) $T(u_n) \rightarrow 0$ in ξ whenever (u_n) is a uniformly bounded sequence in $C_b(X)$ such that $u_n(t) \rightarrow 0$ for every $t \in X$.
- (vi) $T(u_n) \rightarrow 0$ in ξ whenever (u_n) is a uniformly bounded sequence in $C_b(X)$ such that $\text{supp } u_n \cap \text{supp } u_k = \emptyset$ for $n \neq k$.
- (vii) $T(u_n) \rightarrow 0$ in ξ whenever $u_n \rightarrow 0$ in $\sigma(C_b(X), C_b(X)'_{\beta})$.
- (viii) $T(u_n) \rightarrow 0$ in ξ whenever $\sum_{n=1}^{\infty} |\int_X u_n d\mu| < \infty$ for every $\mu \in M(X)$.

Proof. (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) It follows from Theorem 2.4 because by Theorem 3.1 for every $D \in \mathcal{E}$, $\sup_{e' \in D} |\hat{m}_{e'}|(X) < \infty$, where for every $e' \in E'_{\xi}$, $\hat{m}_{e'} \in M(X)$ and $T'(e')(u) = (e' \circ T)(u) = \int_X u d\hat{m}_{e'}$ for $u \in C_b(X)$.

(iii) \Rightarrow (v) Assume that \hat{m} is $\xi_{\mathcal{E}}$ -countably additive. Then for every $D \in \mathcal{E}$, the family $\{|\hat{m}_{e'}| : e' \in D\}$ is uniformly countably additive and since $\sup_{e' \in D} |\hat{m}_{e'}|(X) = c < \infty$ (see Theorem 3.1), there exists $\lambda \in ca(\mathcal{B}_0)^+$ such that $\{|\hat{m}_{e'}| : e' \in D\}$ is uniformly λ -continuous (see [10, Theorem 4, pp. 11–12]), i.e., $\lim_{\lambda(A) \rightarrow 0} \sup_{e' \in D} |\hat{m}_{e'}|(A) = 0$.

Let (u_n) be a sequence in $C_b(X)$ such that $\sup_n \|u_n\|_{\infty} = a < \infty$ and $u_n(t) \rightarrow 0$ for every $t \in X$. Given $\varepsilon > 0$ there exists $\eta > 0$ such that

$\sup_{e' \in D} |\hat{m}_{e'}|(A) \leq \frac{\varepsilon}{2(c+1)}$ whenever $\lambda(A) \leq \eta$, $A \in \mathcal{B}o$. Hence by the Egoroff theorem there exists $A_\eta \in \mathcal{B}o$ with $\lambda(X \setminus A_\eta) \leq \eta$ and $\sup_{t \in A_\eta} |u_n(t)| \leq \frac{\varepsilon}{2(c+1)}$ for $n \geq n_\varepsilon$ and some $n_\varepsilon \in \mathbb{N}$. Then for every $e' \in D$ and $n \geq n_\varepsilon$, we have

$$\begin{aligned} |e'(T(u_n))| &= \left| \int_X u_n d\hat{m}_{e'} \right| \leq \int_X |u_n| d|\hat{m}_{e'}| = \int_{A_\eta} |u_n| d|\hat{m}_{e'}| + \int_{X \setminus A_\eta} |u_n| d|\hat{m}_{e'}| \\ &\leq \frac{\varepsilon}{2(c+1)} |\hat{m}_{e'}|(A_\eta) + a |\hat{m}_{e'}|(X \setminus A_\eta) \leq \frac{\varepsilon}{2(c+1)} c + a \frac{\varepsilon}{2(a+1)} \leq \varepsilon. \end{aligned}$$

Hence $p_D(T(u_n)) \leq \varepsilon$ for $n \geq n_\varepsilon$ and this means that $T(u_n) \rightarrow 0$ in ξ .

(v) \Rightarrow (vi) It is obvious.

(vi) \Rightarrow (ii) Assume that (vi) holds and that (ii) fails to hold, i.e., there exists $D_0 \in \mathcal{E}$ such that $\{\hat{m}_{e'} : e' \in D_0\}$ is not relatively $\sigma(M(X), M(X)')$ -compact. Since $\sup_{e' \in D} |\hat{m}_{e'}|(X) < \infty$ (see Theorem 3.1), according to Theorem 2.4 there exist $\varepsilon_0 > 0$, a pairwise disjoint sequence (O_n) of open sets and a sequence (e'_n) in D_0 such that $|\hat{m}_{e'_n}|(O_n) > \varepsilon_0$. By Lemma 2.2 one can choose a sequence (u_n) in $C_b(X)$ with $\|u_n\|_\infty = 1$ and $\text{supp } u_n \subset O_n$ such that

$$\left| \int_{O_n} u_n d\hat{m}_{e'_n} \right| \geq |\hat{m}_{e'_n}|(O_n) - \frac{\varepsilon_0}{2} > \frac{\varepsilon_0}{2}.$$

Then for $n \in \mathbb{N}$,

$$p_{D_0}(T(u_n)) = \sup \left\{ \left| \int_X u_n d\hat{m}_{e'} \right| : e' \in D_0 \right\} \geq \left| \int_{O_n} u_n d\hat{m}_{e'_n} \right| > \frac{\varepsilon_0}{2}.$$

On the other hand, since $\text{supp } u_n \cap \text{supp } u_k = \emptyset$ for $n \neq k$, we get $p_{D_0}(T(u_n)) \rightarrow 0$. This contradiction establishes that (ii) holds.

(v) \Leftrightarrow (vii) It is obvious.

(v) \Rightarrow (viii) Assume that (v) holds and let (u_n) be a sequence in $C_b(X)$ such that $\sum_{n=1}^\infty \left| \int_X u_n d\mu \right| < \infty$ for every $\mu \in M(X)$. Hence for every $\mu \in M(X)$, $\sup_n \left| \int_X u_n d\mu \right| < \infty$ and it means that the set $\{u_n : n \in \mathbb{N}\}$ is $\sigma(C_b(X), C_b(X)'_\beta)$ -bounded. Hence $\{u_n : n \in \mathbb{N}\}$ is β -bounded, and it follows that $\sup_n \|u_n\|_\infty < \infty$. Moreover, since for every $t \in X$, $\delta_t \in C_b(X)'_\beta$, we get $\sum_{n=1}^\infty |u_n(t)| = \sum_{n=1}^\infty |\delta_t(u_n)| = \sum_{n=1}^\infty \left| \int_X u_n d\mu_t \right| < \infty$ and it follows that $u_n(t) \rightarrow 0$ for every $t \in X$. Thus $T(u_n) \rightarrow 0$ in ξ , i.e., (viii) holds.

(viii) \Rightarrow (vi) Assume that (viii) holds and let (u_n) be a sequence in $C_b(X)$ such that $\sup_n \|u_n\|_\infty = a < \infty$ and $\text{supp } u_n \cap \text{supp } u_k = \emptyset$ for $n \neq k$. Then for every $\mu \in M(X)$, we have

$$\begin{aligned} \sum_{n=1}^\infty \left| \int_X u_n d\mu \right| &\leq \sum_{n=1}^\infty \int_X |u_n| d|\mu| \leq a \sum_{n=1}^\infty |\mu|(\text{supp } u_n) \\ &= a |\mu| \left(\bigcup_{n=1}^\infty \text{supp } u_n \right) \leq a |\mu|(X) < \infty. \end{aligned}$$

Hence $T(u_n) \rightarrow 0$ in ξ , i.e., (vi) holds. □

Assume that $T : C_b(X) \rightarrow E$ is a (β, ξ) -continuous linear operator such that $\hat{T}(B(\mathcal{B}o)) = T''(\pi(B(\mathcal{B}o))) \subset i_E(E)$. From now on let

$$\bar{T}(v) := j_E(\hat{T}(v)) \quad \text{for } v \in B(\mathcal{B}o).$$

Then $\bar{T}(u) = T(u)$ for $u \in C_b(X)$ and the operator $\bar{T} : B(\mathcal{B}o) \rightarrow E$ will be called the *natural extension* of T on $B(\mathcal{B}o)$.

Definition 3.3. A ξ -countably additive measure $m : \mathcal{B}o \rightarrow E$ is called a ξ -Radon measure if m is ξ -regular, i.e., for every $D \in \mathcal{E}$, $A \in \mathcal{B}o$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ with $K \subset A \subset O$ such that $p_D(m(B)) = \sup_{e' \in D} |m_{e'}(B)| \leq \varepsilon$ for every $B \in \mathcal{B}o$ with $B \subset O \setminus K$ (equivalently; $\sup_{e' \in D} |m_{e'}|(O \setminus K) \leq \varepsilon$).

Theorem 3.5. Assume that X is a k -space and (E, ξ) is a sequentially complete lcHs. Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous linear operator and \hat{m} be its representing measure. Assume that $\hat{m}(A) \in i_E(E)$ for every $A \in \mathcal{B}o$ and $m := j_E \circ \hat{m}$. Then the following statements hold:

- (i) $m : \mathcal{B}o \rightarrow E$ is a ξ -Radon measure.
- (ii) $\hat{T}(B(\mathcal{B}o)) \subset i_E(E)$ and $\bar{T}(v) = \int_X v \, dm$ for $v \in B(\mathcal{B}o)$.
- (iii) $\bar{T} : B(\mathcal{B}o) \rightarrow E$ is (β, ξ) -continuous.
- (iv) $T(C_b(X))$ is ξ -dense in $\bar{T}(B(\mathcal{B}o))$.

Proof. (i) Since $m_{e'} = \hat{m}_{e'} \in M(X)$, by the Orlicz-Pettis theorem (see [22, Corollary 1]) m is ξ -countably additive, i.e., for every $D \in \mathcal{E}$, the family $\{m_{e'} : e' \in D\}$ is uniformly countably additive. Hence by Theorem 2.4 $\{|m_{e'}| : e' \in D\}$ is uniformly regular, i.e., m is ξ -regular. This means that m is a ξ -Radon measure.

(ii) Let $v \in B(\mathcal{B}o)$. Choose a sequence (s_n) in $\mathcal{S}(\mathcal{B}o)$ such that $\|v - s_n\|_\infty \rightarrow 0$. Note that $(\int_X s_n \, dm)$ is a ξ -Cauchy sequence in E (see (3.2)) and hence one can define

$$\int_X v \, dm := \xi - \lim \int_X s_n \, dm.$$

Hence using (3.4), we have

$$i_E\left(\int_X v \, dm\right) = \xi_E - i_E\left(\int_X s_n \, dm\right) = \xi_E - \int_X s_n \, d\hat{m} = \int_X v \, d\hat{m} = \hat{T}(v).$$

It follows that $\hat{T}(v) \in i_E(E)$ and hence $\bar{T}(v) = \int_X v \, dm$.

(iii) Since the mapping $j_E : i_E(E) \rightarrow E$ is $(\xi_E|_{i_E(E)}, \xi)$ -continuous, in view of the condition (vi) of Theorem 3.1, $\bar{T} = j_E \circ \hat{T}$ is (β, ξ) -continuous.

(iv) It follows from Corollary 3.3. \square

4. Weakly Compact Operators on $C_b(X)$

Assume that (E, ξ) is a lcHs. Recall that a (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$ is said to be:

- (i) *unconditionally convergent* if the series $\sum_{n=1}^{\infty} T(u_n)$ converges unconditionally in (E, ξ) whenever $\sum_{n=1}^{\infty} |\int_X u_n d\mu| < \infty$ for every $\mu \in M(X)$.
- (ii) *completely continuous* if T maps $\sigma(C_b(X), C_b(X)'_{\beta})$ -Cauchy sequences in $C_b(X)$ onto ξ -convergent sequences in E .
- (iii) *weakly completely continuous* if T maps $\sigma(C_b(X), C_b(X)'_{\beta})$ -Cauchy sequences in $C_b(X)$ onto $\sigma(E, E'_{\xi})$ -convergent sequences in E .

Proposition 4.1. *If $T : C_b(X) \rightarrow E$ is a weakly completely continuous operator, then T is unconditionally convergent.*

Proof. Assume that T is weakly completely continuous. Let (u_n) be a sequence in $C_b(X)$ such that $\sum_{n=1}^{\infty} |\int_X u_n d\mu| < \infty$ for each $\mu \in M(X)$. For a subsequence (u_{k_n}) of (u_n) , let $S_n = \sum_{i=1}^n u_{k_i}$. Then (S_n) is a $\sigma(C_b(X), C_b(X)'_{\beta})$ -Cauchy sequence. It follows that the series $\sum_{n=1}^{\infty} T(u_{k_n})$ is $\sigma(E, E'_{\xi})$ -convergent in E and in view of the Orlicz-Pettis theorem the series $\sum_{n=1}^{\infty} T(u_n)$ is unconditionally converging (see [22, Theorem 1]). This means that T is unconditionally convergent. \square

When X is a compact Hausdorff space and (E, ξ) is a complete lcHs, Grothendieck [15, Theorem 6] gave some necessary and sufficient conditions for (τ_u, ξ) -continuous operator $T : C(X) \rightarrow E$ to be weakly compact. Later, Edwards [12, Theorem 9.4.10] obtained characterizations of weakly compact operators $T : C_o(X) \rightarrow E$, where X is a locally compact Hausdorff space and (E, ξ) is complete. Panchapagesan (see [27, Theorems 2, 3 and 12], [30, Theorem 5.3.7]) has presented equivalent conditions for a (τ_u, ξ) -continuous operator $T : C_o(X) \rightarrow E$ to be weakly compact if X is a locally compact Hausdorff space and (E, ξ) is quasicomplete. Now using the results of Sect. 3 we present equivalent conditions for a (β, ξ) -continuous operator $T : C_b(X) \rightarrow E$ to be weakly compact, where X is a k -space and (E, ξ) is quasicomplete.

Theorem 4.2. *Assume that X is a k -space and (E, ξ) is a quasicomplete lcHs. Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous linear operator and \hat{m} be its representing measure. Then the following statements are equivalent:*

- (i) T is weakly compact.
- (ii) $T''(\pi(B(\mathcal{B}o))) \subset i_E(E)$.
- (iii) $\hat{m}(A) \in i_E(E)$ for every $A \in \mathcal{B}o$.
- (iv) $\hat{m}(O) \in i_E(E)$ for every $O \in \mathcal{T}$.
- (v) For every $D \in \mathcal{E}$, the set $\{\hat{m}_{e'} : e' \in D\}$ is relatively $\sigma(M(X), M(X)')$ -compact.
- (vi) $\hat{m} : \mathcal{B}o \rightarrow E''_{\xi}$ is $\xi_{\mathcal{E}}$ -countably additive.
- (vii) $\hat{m} : \mathcal{B}o \rightarrow E''_{\xi}$ is $\xi_{\mathcal{E}}$ -regular.

- (viii) $T(u_n) \rightarrow 0$ in ξ whenever (u_n) is a uniformly bounded sequence in $C_b(X)$ such that $u_n(t) \rightarrow 0$ for every $t \in X$.
- (ix) $T(u_n) \rightarrow 0$ in ξ whenever (u_n) is a uniformly bounded sequence in $C_b(X)$ such that $\text{supp } u_n \cap \text{supp } u_k = \emptyset$ for $n \neq k$.
- (x) $T(u_n) \rightarrow 0$ in ξ whenever $u_n \rightarrow 0$ in $\sigma(C_b(X), C_b(X)'_\beta)$.
- (xi) $T(u_n) \rightarrow 0$ in ξ whenever $\sum_{n=1}^{\infty} |\int_X u_n d\mu| < \infty$ for every $\mu \in M(X)$.
- (xii) T is completely continuous.
- (xiii) T is unconditionally convergent.
- (xiv) T is weakly completely continuous.

Proof. (i) \Rightarrow (ii) Assume that T is weakly compact. Then $T''(C_b(X)''_\beta) \subset i_E(E)$ (see [12, Corollary 9.3.2]) and it follows that $T''(\pi(B(\mathcal{B}o))) \subset i_E(E)$.

(ii) \Rightarrow (iii) \Rightarrow (iv) It is obvious.

(iv) \Rightarrow (v) Assume that $\hat{m}(O) \in i_E(E)$ for every $O \in \mathcal{T}$ and $D \in \mathcal{E}$. Let (O_n) be a pairwise disjoint sequence in \mathcal{T} and $O = \bigcup_{n=1}^{\infty} O_n$. Since for every $e' \in E'_\xi$, $\hat{m}_{e'} \in M(X)$ (see Theorem 3.1), we get

$$e'(j_E(\hat{m}(O))) = \hat{m}_{e'}(O) = \sum_{n=1}^{\infty} \hat{m}_{e'}(O_n) = \sum_{n=1}^{\infty} e'(j_E(\hat{m}(O_n))).$$

Then by the Orlicz-Pettis theorem (see [22, Theorem 1]), we get $j_E(\hat{m}(O)) = \sum_{n=1}^{\infty} j_E(\hat{m}(O_n))$ in (E, ξ) . Hence

$$p_D(j_E(\hat{m}(O_n))) = \sup\{|e'(j_E(\hat{m}(O_n)))| : e' \in D\} = \sup\{|\hat{m}_{e'}(O_n)| : e' \in D\} \rightarrow 0$$

and by Theorem 2.4, $\{\hat{m}_{e'} : e' \in D\}$ is relatively $\sigma(M(X), M(X)')$ -compact.

(v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii) \Leftrightarrow (ix) \Leftrightarrow (x) \Leftrightarrow (xi) See Corollary 3.4.

(vi) \Rightarrow (i) It follows from [12, Corollary 9.3.2].

(iii) \Rightarrow (xii) Assume that (iii) holds. Then $m = j_E \circ \hat{m} : \mathcal{B}o \rightarrow E$ is ξ -countably additive because $m_{e'} = \hat{m}_{e'} \in M(X)$. Assume that (u_n) is a $\sigma(C_b(X), C_b(X)'_\beta)$ -Cauchy sequence in $C_b(X)$. Then the set $\{u_n : n \in \mathbb{N}\}$ is β -bounded, so $\sup_n \|u_n\|_\infty < \infty$. It follows that $\lim \delta_t(u_n) = \lim u_n(t) = v_o(t)$ exists for every $t \in X$, and hence $v_o \in B(\mathcal{B}o)$. Then by the Lebesgue bounded convergence theorem (see [27, Proposition 7]) $T(u_n) = \int_X u_n dm \rightarrow \int_X v_o dm$ in ξ .

(xiii) \Rightarrow (xi) It is obvious.

(iii) \Rightarrow (xiii) Assume that (iii) holds. Then $m = j_E \circ \hat{m} : \mathcal{B}o \rightarrow E$ is ξ -countable additive because $m_{e'} = \hat{m}_{e'} \in M(X)$. Assume that $\sum_{n=1}^{\infty} |\int_X u_n d\mu| < \infty$ for every $\mu \in M(X)$. Hence $\sum_{n=1}^{\infty} |u_n(t)| < \infty$ for every $t \in X$ because $\delta_t \in C_b(X)'_\beta$. Let $S_n(t) = \sum_{i=1}^n u_i(t)$ for $t \in X$. Then (S_n) is a $\sigma(C_b(X), C_b(X)'_\beta)$ -bounded sequence and it follows that (S_n) is β -bounded and hence $\sup_n \|S_n\|_\infty < \infty$. Let $v_o(t) = \lim_n S_n(t)$ for $t \in X$. Then $v_o \in B(\mathcal{B}o)$ and by the Lebesgue bounded convergence theorem (see [27, Proposition 7]) and Theorem 3.5, we have

$$\lim_n \sum_{i=1}^n T(u_i) = \lim_n \int_X S_n dm = \int_X v_o dm \text{ in } (E, \xi).$$

Finally, if (n_j) is any permutation of \mathbb{N} , then $\lim_n \sum_{j=1}^n u_{n_j}(t) = v_o(t)$ for $t \in X$. Then $\sum_{j=1}^\infty T(u_{n_j}) = \int_X v_o dm$, as desired.

(xii) \Rightarrow (xiv) It is obvious.

(xiv) \Rightarrow (xiii) See Proposition 4.1. \square

Corollary 4.3. *Assume that X is a k -space and (E, ξ) is a quasicomplete $lCHs$. Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous weakly compact operator and \hat{m} be its representing measure. Then the following statements hold:*

- (i) $m = j_E \circ \hat{m} : \mathcal{B}_o \rightarrow E$ is a ξ -Radon measure.
- (ii) The extension operator $\bar{T} : B(\mathcal{B}_o) \rightarrow E$ is (β, ξ) -continuous and weakly compact.
- (iii) $\bar{T}(v_n) \rightarrow 0$ in ξ whenever (v_n) is a uniformly bounded sequence in $B(\mathcal{B}_o)$ such that $v_n(t) \rightarrow 0$ for every $t \in X$.
- (iv) $\bar{T}(v_\alpha) \rightarrow 0$ in ξ whenever (v_α) is a uniformly bounded net in $B(\mathcal{B}_o)$ such that $v_\alpha \rightarrow 0$ in τ_c .

Proof. (i) It follows from Theorems 4.2 and 3.5.

- (ii) By Theorems 4.2 and 3.5 \bar{T} is (β, ξ) -continuous and since $m := j_E \circ \hat{m}$ is ξ -countably additive, \bar{T} is weakly compact (see [26, Theorem 1]).
- (iii) It follows from the Lebesgue bounded convergence theorem (see [27, Proposition 7]).
- (iv) It follows from (ii) because β and τ_c agree on uniformly bounded sets in $B(\mathcal{B}_o)$. \square

For definitions and details concerning the strict Dunford-Pettis property, the Dunford-Pettis property and the Dieudonné property, we refer a reader to [12, § 9.4]. As an application of Theorem 4.2 we get:

Corollary 4.4. *Assume that X is a k -space. Then the space $(C_b(X), \beta)$ has both the strict Dunford-Pettis property and the Dieudonné property.*

Remark 4.1. The fact that the space $(C_b(X), \beta)$ has the strict Dunford-Pettis property was proved in a different way by Aguayo and Sanchez [1, Theorem 2.4]. Moreover, Khurana [19, Theorem 3] and Chacoń and Vielma [5, Theorem 3.1] showed that $(C_b(X), \beta)$ has the Dunford-Pettis property.

Corollary 4.5. *Assume that X is a k -space and (E, ξ) is a quasicomplete $lCHs$. Let $T : C_b(X) \rightarrow E$ be a (β, ξ) -continuous weakly compact operator. Then*

- (i) T maps relatively $\sigma(C_b(X), M(X))$ -countably compact sets in $C_b(X)$ onto relatively ξ -compact sets in E .
- (ii) T maps uniformly bounded relatively τ_p -sequentially compact sets in $C_b(X)$ onto relatively ξ -compact sets in E (here τ_p denotes the pointwise convergence topology on $C_b(X)$).

- Proof.* (i) Let H be a relatively $\sigma(C_b(X), C_b(X)'_\beta)$ -countably compact subset of $C_b(X)$. Then the closed absolutely convex hull of H is $\sigma(C_b(X), C_b(X)'_\beta)$ -compact (see [H₂, Theorem 4]). Since the space $(C_b(X), \beta)$ has the Dunford-Pettis property (see [19, Theorem 3]), we obtain that $T(H)$ is relatively ξ -compact.
- (ii) Let H be a uniformly bounded relatively τ_p -sequentially compact subset of $C_b(X)$. Then by the Lebesgue dominated convergence theorem, H is relatively $\sigma(C_b(X), C_b(X)'_\beta)$ -sequentially compact (see [H₂, Proposition 2]) and it follows that H is relatively $\sigma(C_b(X), C_b(X)'_\beta)$ -countably compact. Hence in view of (i) $T(H)$ is relatively ξ -compact. \square

Grothendieck [15] proved that if X is a compact Hausdorff space and E is a weakly sequentially complete Banach space, then every bounded operator $T : C(X) \rightarrow E$ is weakly compact. Panchapagesan has shown that if X is a locally compact Hausdorff space and (E, ξ) is quasicomplete and contains no isomorphic copy of c_o , then every (τ_u, ξ) -continuous linear operator $T : C_o(X) \rightarrow E$ is weakly compact (see [29, Corollary 2]). Now we extend this result to operators on $C_b(X)$ in the setting of k -spaces.

Corollary 4.6. *Assume that X is a k -space and (E, ξ) is a quasicomplete $lcHs$ that contains no isomorphic copy of c_o . Then every (β, ξ) -continuous linear operator $T : C_b(X) \rightarrow E$ is weakly compact.*

Proof. Let \hat{m} be the representing measure of T . Then $\hat{m}_{e'} \in M(X)$ for every $e' \in E'_\xi$ (see Theorem 3.1). Assume that (u_n) is a sequence in $C_b(X)$ such that $\sum_{n=1}^\infty |\int_X u_n d\mu| < \infty$ for every $\mu \in M(X)$. Hence for every $e' \in E'_\xi$ by Theorem 3.1, we have

$$\sum_{n=1}^\infty |e'(T(u_n))| = \sum_{n=1}^\infty \left| \int_X u_n d\hat{m}_{e'} \right| < \infty.$$

Since E contains no isomorphic copy of c_o , by [37, Theorem 4] $\sum_{n=1}^\infty T(u_n)$ converges unconditionally in (E, ξ) and it means that T is unconditionally convergent. Hence, by Theorem 4.2 T is weakly compact. \square

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Aguayo, J., Sánchez, J.: The Dunford-Pettis property on vector-valued continuous and bounded functions. *Bull. Aust. Math. Soc.* **48**, 303–311 (1993)
- [2] Bartle, R., Dunford, N., Schwartz, J.: Weak compactness and vector measures. *Can. J. Math.* **7**, 189–305 (1955)
- [3] Brooks, J.K.: On a theorem of Dieudonné. *Adv. Math.* **36**, 165–168 (1980)
- [4] Buck, R.C.: Bounded continuous functions on a locally compact space. *Mich. Math. J.* **5**, 95–104 (1958)
- [5] Chacón, G., Vielma, J.: The Dunford-Pettis property and strict topologies. *Sci. Math.* **2**(1), 99–105 (1999)
- [6] Conway, J.B.: The strict topology and compactness in the space of measures, II. *Trans. Am. Math. Soc.* **126**, 474–486 (1967)
- [7] Cooper, J.B.: The strict topology and spaces with mixed topologies. *Proc. Am. Math. Soc.* **30**(3), 583–592 (1971)
- [8] Dieudonné, J.: Sur la convergence de suites de mesures de Radon. *An. Acad. Brasil. Ci.* **23**(21–38), 277–282 (1951)
- [9] Diestel, J.: *Sequences and Series in Banach Spaces*, Graduate Texts in Math. Springer-Verlag, Berlin (1984)
- [10] Diestel, J., Uhl, J.J.: *Vector measures*, Amer. Math. Soc., Math. Surveys 15, Providence, RI, (1977)
- [11] Dinculeanu, N., Klavanek, I.: On vector measures. *Proc. Lond. Math. Soc.* **17**(3), 505–512 (1967)
- [12] Edwards, R.E.: *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, New York (1965)
- [13] Engelking, R.: *General Topology*. Heldermann Verlag, Berlin (1989)
- [14] Giles, R.: A generalization of the strict topology. *Trans. Am. Math. Soc.* **161**, 467–474 (1971)
- [15] Grothendieck, A.: Sur les applications lineaires faiblement compactes d'espaces de type $C(K)$. *Can. J. Math.* **5**, 129–173 (1953)
- [16] Hoffmann-Jørgensen, J.: Vector measures. *Math. Scand.* **28**, 5–32 (1971)
- [17] Hoffmann-Jørgensen, J.: A generalization of the strict topology. *Math. Scand.* **30**, 313–323 (1972)
- [18] Kantorovitch, L.V., Akilov, G.P.: *Functional Analysis*. Pergamon Press, Oxford-Elmsford, New York (1982)
- [19] Khurana, S.S.: Dunford-Pettis property. *J. Math. Anal. Appl.* **65**, 361–364 (1978)
- [20] Khurana, S.S., Vielma, J.: Weak sequential convergence and weak compactness in spaces of vector-valued continuous functions. *J. Math. Anal. Appl.* **195**, 251–260 (1995)
- [21] Lewis, D.R.: Integration with respect to vector measures. *Pac. J. Math.* **33**(1), 157–165 (1970)
- [22] McArthur, C.: On the theorem of Orlicz and Pettis. *Pac. J. Math.* **22**(2), 297–302 (1967)

- [23] Nowak, M.: Linear operators on the space of bounded continuous functions with strict topologies. *Positivity* **14**, 831–839 (2010)
- [24] Nowak, M.: Vector measures and strict topologies. *Topol. Appl.* **159**(5), 1421–1432 (2012)
- [25] Nowak, M.: A Riesz representation theory for completely regular Hausdorff spaces and its applications. *Open Math.* **14**, 474–496 (2016)
- [26] Panchapagesan, T.V.: Applications of a theorem of Grothendieck to vector measures. *J. Math. Anal. Appl.* **214**, 89–101 (1997)
- [27] Panchapagesan, T.V.: Characterization of weakly compact operators on $C_0(T)$. *Trans. Am. Math. Soc.* **350**(12), 4849–4867 (1998)
- [28] Panchapagesan, T.V.: Baire and σ -Borel characterizations of weakly compact sets in $M(T)$. *Trans. Am. Math. Soc.* **350**(12), 4839–4847 (1998)
- [29] Panchapagesan, T.V.: Weak compactness of unconditionally convergent operators on $C_0(T)$. *Math. Slovaca* **52**(1), 57–66 (2002)
- [30] Panchapagesan, T.V.: The Bartle–Dunford–Schwartz Integral, *Mathematical Monographs (New Series)*, 69. Birkhäuser Verlag, Basel (2008)
- [31] Robertson, A.P., Robertson, W.: *Topological Vector Spaces*. Cambridge University Press, Cambridge (1964)
- [32] Schmets, J., Zafarani, J.: Strict topologies and (gDF)-spaces. *Arch. Math.* **49**, 227–231 (1987)
- [33] Sentilles, F.D.: Bounded continuous functions on a completely regular space. *Trans. Am. Math. Soc.* **168**, 311–336 (1972)
- [34] Shuchat, A.: Vector measures and scalar operators in locally convex spaces. *Mich. Math. J.* **24**, 303–309 (1977)
- [35] Siggini, K.K.: Compactness and convergence of set-valued measures. *Colloq. Math.* **114**(2), 177–189 (2009)
- [36] Topsoe, F.: Compactness in spaces of measures. *Stud. Math.* **36**, 195–212 (1970)
- [37] Tumarkin, J.B.: Locally convex spaces with basis. *Dokl. Acad. Nauk SSSR* **195**, 1278–1281 (1970). (English Transl., *Soviet Math. Dokl.* 11 (1970), 1672–1675)
- [38] Wiweger, A.: Linear spaces with mixed topology. *Stud. Math.* **20**, 47–68 (1961)

Marian Nowak
Faculty of Mathematics
Computer Science and Econometrics
University of Zielona Góra
Ul. Szafrana 4A
65-516 Zielona Góra
Poland
e-mail: M.Nowak@wmie.uz.zgora.pl

Received: December 10, 2016.

Accepted: March 24, 2017.